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COMMENT

Note on Bäcklund transformations

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Abstract. The method of obtaining Bäcklund transformations proposed by Chern and Tenenblat fits completely the approach of obtaining Bäcklund transformations by prolongation techniques. For $\kappa\delta\nu$, $\mu\kappa\delta\nu$ and sine-Gordon equations the only difference consists in the application of a non-linear representation of the prolongation algebra other than the usual one. This representation can be obtained by a coordinate transformation of the prolongation variable.

In the literature there are a number of methods available for obtaining Bäcklund transformations for integrable systems or integrable evolution equations.

An intriguing, and in our opinion, very important aspect of these techniques is the study and understanding of their connection and common background with the aim of achieving some unification of these methods. It is the purpose of this comment to contribute to this understanding.

In a recent paper [1] Chern and Tenenblat investigate properties of evolution equations in terms of pseudospherical surfaces. Moreover they discuss a geometrical result which, under certain conditions, provides Bäcklund transformations for differential equations which describe pseudospherical surfaces. The relation between non-linear partial differential equations and geometric properties of surfaces dates back to the nineteenth century and was initiated by Bianchi, Bäcklund and Darboux. They have shown that, for example, the sine-Gordon equation can be interpreted in terms of transformations of pseudospherical surfaces. Later this result was considerably extended by, for example, Sasaki [2].

Wahlquist and Estabrook [3, 4] study evolution equations in connection with prolongation structures and furnish a way to obtain Bäcklund transformations from this prolongation structure. We remark that in [1] by definition a non-linear differential equation describes a pseudospherical surface precisely when the equation admits a $sl(2)$ prolongation structure. From this observation it follows that the method of obtaining Bäcklund transformations proposed by Chern and Tenenblat fits completely within the prolongation approach. We shall show that both the results and the derivation of these results for the three concrete cases, the $\kappa\delta\nu$, $\mu\kappa\delta\nu$ and sine-Gordon equations, are closely related to well known derivations and results which are already available in the literature. In fact we shall see that for these three examples, apart from a very elegant geometrical interpretation, the only difference consists of application of another non-linear representation of the prolongation algebra than the usual one. This representation can be obtained by a coordinate transformation of the prolongation variable.

The $\mathfrak{sl}(2)$ -valued prolongation form for the κ_{dv} equation is given, see e.g. [3, 5] by

$$\omega = F dx + G dt \quad (1)$$

with

$$\begin{aligned} F &= -e + (2u - \lambda)f \\ G &= 4(u + \lambda)e + 2(-u_{xx} - 4u^2 + 2\lambda^2 - 2u\lambda)f + 2u_x h \end{aligned} \quad (2)$$

where e, f, h is the usual basis of $\mathfrak{sl}(2)$ such that

$$[h, e] = 2e \quad [h, f] = -2f \quad [e, f] = h$$

where λ is a parameter. Equating to zero the curvature $d\omega - \omega \wedge \omega$ equals the κ_{dv} equation

$$u_t + u_{xxx} + 12uu_x = 0. \quad (3)$$

The corresponding 1-form for this equation considered by Chern and Tenenblat is

$$\omega = \omega_1 X_1 + \omega_2 X_2 + \omega_3 X_3 \quad (4)$$

where

$$\begin{aligned} \omega_1 &= (1 - 2u) dx + (2u_{xx} - 2\eta u_x + 2\eta^2 u + 8u^2 - \eta^2 - 4u) dt \\ \omega_2 &= \eta dx + (-\eta^3 - 4\eta u + 4u_x) dt \end{aligned} \quad (5)$$

$$\omega_3 = -(1 + 2u) dx + (2u_{xx} - 2\eta u_x + 2\eta^2 u + 8u^2 + \eta^2 + 4u)$$

and the basis X_1, X_2, X_3 of $\mathfrak{sl}(2)$ satisfies

$$[X_1, X_2] = X_3 \quad [X_2, X_3] = -X_1 \quad [X_1, X_3] = X_2.$$

Comparing (1) and (4) we find

$$\begin{aligned} e &= (-1 + \frac{1}{4}\eta^2)X_1 - \eta X_2 + (1 + \frac{1}{4}\eta^2)X_3 \\ f &= -X_1 - X_3 \\ h &= -\eta X_1 + 2X_2 - \eta X_3 \end{aligned} \quad (6)$$

and

$$\eta^2 = 4\lambda.$$

Wahlquist and Estabrook gave a realisation of (1) applying the non-linear representation of $\mathfrak{sl}(2)$ in the form

$$e = -y^2 \quad f = 1 \quad h = 2y \quad (7)$$

with the usual bracket for functions of the variable y . In this case the condition $\omega = 0$ is equivalent to

$$dy + (2u + y^2 - \lambda) dx - 4[(u + \lambda)(2u + y^2 - \lambda) + \frac{1}{2}u_{xx} - u_x y] dt = 0. \quad (8)$$

Chern and Tenenblat use the non-linear representation

$$X_1 = -\sin \phi \quad X_2 = -\cos \phi \quad X_3 = -1 \quad (9)$$

functions of the variable ϕ .

In this realisation $\omega = 0$ is equivalent to

$$\begin{aligned} d\phi + [(1 + 2u) - (1 - 2u) \sin \phi - \eta \cos \phi] dx - [(1 + \sin \phi)(2u_{xx} - 2\eta u_x + 2\eta^2 u + 8u^2) \\ + (1 - \sin \phi)(\eta^2 + 4u) + \cos \phi(-\eta^3 - 4\eta u + 4u_x)] dt = 0. \end{aligned} \quad (10)$$

Using (8) Wahlquist and Estabrook obtained the self-Bäcklund transformation for the $\kappa\Delta v$ equation

$$\bar{u} = -u - y^2 + \lambda \tag{11}$$

where y has to satisfy (8) and \bar{u} is a second solution of the $\kappa\Delta v$ equation (3).

Chern and Tenenblat, using (9), derived the same Bäcklund transformation in the form

$$\bar{u} = u + \frac{\phi_x}{1 + \sin \phi} = -u - \frac{1 - \sin \phi - \eta \cos \phi}{1 + \sin \phi}. \tag{12}$$

A simple calculation shows that the coordinate transformation

$$y = \frac{\cos \phi}{1 + \sin \phi} - \frac{1}{2}\eta \tag{13}$$

transforms the non-linear representation (7) of $sl(2)$, used by Wahlquist and Estabrook, into the non-linear representation (9) and the prolongation (8) into the form (10). Another very simple calculation shows that the right-hand side of (11) equals the right-hand side of (12). So, apart from some very elegant geometric interpretation, the method of Chern and Tenenblat for obtaining the Bäcklund transformation for the $\kappa\Delta v$ equation differs from the method used by Wahlquist and Estabrook only by a coordinate transformation.

The $sl(2)$ prolongation for the $m\kappa\Delta v$ equation is given by (see [4])

$$\omega = F dx + G dt \tag{14}$$

with

$$\begin{aligned} F &= \lambda h + u(e - f) \\ G &= -(p + 2u^3 - 4u\lambda^2)(e - f) - 2\lambda(2\lambda^2 + u^2)h - 2u_x\lambda(e + f). \end{aligned} \tag{15}$$

In this case, the zero curvature condition $d\omega - \omega \wedge \omega = 0$ is equivalent to the $m\kappa\Delta v$ equation

$$u_t + u_{xxx} + 6u^2u_x = 0. \tag{16}$$

With respect to this non-linear evolution equation Chern and Tenenblat consider the form

$$\omega = \omega_1 X_1 + \omega_2 X_2 + \omega_3 X_3 \tag{17}$$

$$\omega_1 = 2\eta u_x dt$$

$$\omega_2 = \eta dx - (2\eta u^2 + \eta^3) dt \tag{18}$$

$$\omega_3 = 2u dx - (2u_{xx} + 4u^3 + 2\eta^2 u) dt.$$

Setting

$$e = -X_1 + X_3 \quad f = -X_1 - X_3 \quad h = 2X_2 \quad \eta = 2\lambda \tag{19}$$

we have equality of (14) and (17).

Applying again the non-linear representation (7) we obtain the prolongation

$$\begin{aligned} \omega &= dy + [2\lambda y - u(y^2 + 1)] dx + [-4\lambda(2\lambda^2 + u^2)y \\ &\quad + (u_{xx} + 2u^3 + 4\lambda^2 u)(y^2 + 1) + 2\lambda u_x(y^2 - 1)] dt = 0. \end{aligned} \tag{20}$$

The non-linear representation (9) applied in [1] gives

$$\omega = d\phi - (\eta \cos \phi + 2u) dx - [2\eta u_x \sin \phi - (2\eta u^2 + \eta^3) \cos \phi - 2(u_{xx} + 2u^3 + \eta^2 u)] dt = 0. \quad (21)$$

In this case the coordinate transformation

$$y = \tan\left(\frac{1}{2}\phi - \frac{1}{4}\pi\right) \quad (22)$$

transforms the non-linear representation (7) of $sl(2)$ into the representation (9) and the prolongation (20) into (21). The derivation of the Bäcklund transformation in [1], based on equation (21), proceeds essentially along the same lines as the derivation of the Bäcklund transformation given by Chen [6].

Finally we discuss the sine-Gordon equation

$$u_{xt} = \sin u.$$

Using the realisation (7), the $sl(2)$ prolongation for the sine-Gordon equation is given in [7]. In our notation it is described by

$$\omega = dy + \left[\frac{1}{2}(1 + y^2)u_x - (1/\lambda)y\right] dx - \lambda\left[\frac{1}{2}(1 - y^2) \sin u + y \cos u\right] dt = 0. \quad (23)$$

By the same coordinate transformation (22) it transforms into

$$\omega = d\phi + [u_x + (1/\lambda) \cos \phi] dx + \lambda \cos(u - \phi) dt = 0$$

which is the same as the form in [1] if we put $\eta = 1/\lambda$.

References

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